Spatiotemporally periodic patterns in symmetrically coupled map lattices

Qu Zhilin
Department of Physics, Beijing Normal University, Beijing 100875, China

Hu Gang
China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080 China;
Department of Physics, Beijing Normal University, Beijing 100875, China;
and Institute of Theoretical Physics, Academia Sinica, Beijing, China

Ma Benkun and Tian Gang
Department of Physics, Beijing Normal University, Beijing 100875, China
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We deduce a class of spatiotemporal periodic orbits in locally to globally coupled map lattices, from the known orbits with smaller phase spaces, without analytically and numerically solving the modeling equations. The stability of the deduced orbits is investigated and we can reduce the problem to analyze much smaller matrices corresponding to the building block of their spatial periodicity or to the building block of the spatial periodicity of the original orbits from which we construct the new orbits. In the two-dimensional case the problem is considerably simplified.

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I. INTRODUCTION

Coupled nonlinear systems have recently activated great interest and various studies have been carried out on them [1–23]. The investigation mainly focuses on two directions. One is coupled map lattices (CMLs) [1–16], which are coupled by discrete maps and possess the advantage of being easy to handle analytically and numerically. The other is the coupled nonlinear oscillators [17–23], which are coupled by a set of ordinary differential equations and can model systems exactly.

The CML is more idealized than realistic. But the study of CMLs can give a deep understanding of the basic properties of systems with many degrees of freedom. A rich spectrum of universality classes has been found that includes frozen random patterns, pattern selections with suppression of chaos, spatiotemporal intermittency, supertransients, traveling waves, etc. It has been used to model the phenomena in spatially extended systems such as turbulence in hydrodynamics, spatiotemporal intermittency and spiral waves in real-life systems [10,11], and crystal growth [12]. It has also been used to model the evolution of genetic sequences in biology [13] and the spatiotemporal behavior of chemical reactions [14]. The CML model has been successful in modeling the dynamics in a computationally more efficient manner; for example, it has been developed as an efficient scheme of simulating the kinetics of important equations in phase-ordering processes such as Cahn-Hilliard and time-dependent Ginzburg-Landau equations [15,16]. Furthermore, some physical experiments have also been carried out on coupled nonlinear oscillators of which the results are in good agreement with the CML models [17].

Coupled nonlinear oscillators are much more realistic than CMLs, but more difficult in a mathematical aspect. Practical examples are coupled *p-n* junctions [17], coupled pendulums [18], coupled Josephson-junction arrays [19], electronic oscillator circuits [20], charge-density waves [21], coupled laser arrays [22], multimode lasers [23], and some chemical reaction systems [11]. These systems have the great potential of practical applications. Many interesting phenomena have been revealed theoretically and experimentally in recent investigations, such as splay-phase states, noise sensitivity, spatiotemporal intermittency, attractor crowding, etc. Thus an understanding of the dynamical properties of these systems is obviously of theoretical and experimental importance.

In this paper we will be dealing with a special class of spatially periodic states in symmetrically coupled nonlinear systems by concretely employing CML models. Similar investigations can be carried out in systems of coupled nonlinear oscillators. We first predict a special class of spatially periodic orbits from the spatial period-2 orbits in the one-dimensional CML, and the stability of the predicted orbits is discussed in Sec. II. In Sec. III some examples are presented and discussed to demonstrate our arguments in Sec. II. In Sec. IV we deduce a class of spatially periodic orbits in the two-dimensional CML from the ones existing in the one-dimensional CML, and we demonstrate that the analysis of their stability can be reduced to the problems of analyzing that of the one-dimensional building blocks.

II. ONE-DIMENSIONAL CASE

In this section we address the problems of how to deduce a spatially high periodic orbit from a low periodic one in a CML according to its symmetry and how its stability relates to the low periodic one. We consider, for instance, the following symmetrical CML model:

$$x_{n+1}(i)=f[x_n(i)]+\frac{\epsilon}{2s}\sum_{\substack{j=-s\\ j\neq 0}}^{s}\{g[x_n(i+j)]-g]x_n(i)]\}\ , \tag{1}$$

where $f(x)$ and $g(x)$ are some maps such as a tent map, logistic map, circle map, etc.; $x_n(i)$ is the variable associated with the ith lattice site at the nth iteration; ϵ represents the diffusion constant and s the coupling length. Periodic boundaries are always assumed, i.e., $x_n(i)=x_n(i+L)$ with L being the system size. The CML (1) is degenerate if $f(x)=g(x)$. If $s=L-1$, the CML (1) is a globally coupled map lattice (GCML), which has been widely investigated by Kaneko [3].

In order to deduce spatially high periodic orbits for the CML (1) we first consider the most widely studied CML model [1,2,5–8]

$$x_{n+1}(i)=f[x_n(i)]+\frac{\epsilon}{2}\{g[x_n(i-1)]+g[x_n(i+1)]-2g[x_n(i)]\}\ . \tag{2}$$

Many inhomogeneous periodic orbits can be obtained from the CML (2) analytically and numerically [5–8]. Here we pay attention to the spatial period-2 orbits for our special purposes. [In this paper we denote the spatiotemporally periodic orbits in the one-dimensional CML $S_M(k,s,R)$, where M is the temporal periodicity, k the spatial periodicity, s the coupling length, and R the replicas of the spatial periodicity, i.e., $R=L/k$.] Assuming an $S_M(2,1,R)$ orbit, $(y_m z_m y_m z_m \cdots y_m z_m)$, $m=1,2,\ldots,M$, exists in the CML (2), where we define $y_m=y_m(\epsilon)$ and $z_m=z_m(\epsilon)$ as functions of ϵ only. Inserting y_m and z_m into Eq. (2) we obtain

$$z_{m+1}=f(z_m)+\epsilon[g(y_m)-g(z_m)]\ . \tag{3}$$

Instead of solving Eq. (2) directly we can deduce, from the $S_M(2,1,R)$ orbit, an $S_M(4,1,R)$ orbit. Assuming an $S_M(4,1,R)$ orbit, $(y'_m y'_m z'_m z'_m y'_m y'_m z'_m z'_m \cdots y'_m y'_m z'_m z'_m)$, $m=1,2,\ldots,M$, exists in the CML (2) and provided $x_n(i)=x_n(i+1)=y'_m\neq x_n(i-1)=z'_m$, we obtain that y'_m and z'_m satisfy

$$z'_{m+1}=f(z'_m)+\frac{\epsilon}{2}[g(y'_m)-g(z'_m)]\ . \tag{4}$$

Comparing Eq. (3) with Eq. (4), we find that if we substitute ϵ by 2ϵ in Eq. (4) we can obtain at once that $y'_m=y_m$ and $z'_m=z_m$, or in another way, $y'_m(\epsilon)=y_m(\epsilon/2)$ and $z'_m(\epsilon)=z_m(\epsilon/2)$. Thus we can conclude that if there exists an $S_M(2,1,R)$ orbit at $\epsilon=\epsilon_0$ in the CML (2), there must be an $S_M(4,1,R)$ orbit at $\epsilon=2\epsilon_0$ (here we imply that the system size is not fixed but can be changed to satisfy the condition for the orbit to exist, and this assumption is maintained throughout all our investigations in this paper). We cannot construct other similar orbits in the CML (2) from the $S_M(2,1,R)$ orbit. But for the CML (1) we can construct a series of such orbits [denoted as $S_M(2N,s,R)$],

$$(\overbrace{y_m y_m \cdots y_m}^{N}\overbrace{z_m z_m \cdots z_m}^{N} y_m \cdots)\ , \tag{5}$$

from the $S_M(2,1,R)$ orbit with y_m and z_m satisfying

$$z_{m+1}=f(z_m)+\beta\epsilon[g(y_m)-g(z_m)]\ , \tag{6}$$

where β is the scaling factor of the diffusion constant, which has two possible values $\frac{1}{2}$ and $(s+1)/2s$, and the state variables are $y_m=y_m(\beta\epsilon)$. Equation (6) means that if there exists an $S_M(2,1,R)$ orbit at $\epsilon=\epsilon_0$ in the CML (2), there may be an $S_M(2N,s,R)$ orbit at $\epsilon=\epsilon_0/\beta$ in the CML (1). With careful study we find that β generally satisfies

$$\beta=\begin{cases}(\text{nonexistent})\ , & s=kN+1\ ,\quad i=1,2,\ldots,N-2\\ \frac{1}{2}\ , & s=(2k+1)N-1\\ \dfrac{s+1}{2s}\ , & s=(2k+1)N\\ \dfrac{s+1}{2s}\ , & s=(2k+2)N-1\\ \frac{1}{2}\ , & s=(2k+2)N\end{cases} \tag{7a}$$

(where $k=0,1,2,3,\ldots$) for $S_M(2N,s,R)$ orbits. For $S_M(2,s,R)$ orbits, β simply satisfies

$$\beta=\begin{cases}\dfrac{s+1}{2s} & (\text{odd } s)\\ \frac{1}{2} & (\text{even } s)\ .\end{cases} \tag{7b}$$

An exception that is not included in Eqs. (7) is the case of $s=L-1$, i.e., the case of the GCML. It is very interesting that in this case any spatial configuration of the states y_m and z_m is a solution of Eq. (1) due to its symmetry, only requiring that the total numbers of the states y_m and z_m are equal. Therefore, if there exist $S_M(2,1,R)$ orbits, it implies that there are $L!/[(L/2)!]^2$ orbits existing in this case, including all possible spatially periodic orbits $S_M(2N,L-1,R)$ (i.e., N can range from 1 to $L/2$) and frozen random patterns. And all the orbits in the GCML have $\beta=L/2(L-1)$ no matter how the spatial configuration is. Therefore, all orbits belonging to this special class are predicted from the spatial period-2 orbit for the locally to globally coupled map lattices.

The study of the periodic orbits in the CMLs has been carried out by several authors [4–7]. Amritkar *et al.* [5] have given a general discussion about how to analyze the stability of a spatially periodic state and have reduced the problem of analyzing the stability of a spatially periodic orbit to that of analyzing matrices corresponding to the building blocks of spatial periodicity. In this study we construct spatially periodic orbits from the $S_M(2,1,R)$ orbit. A problem arising is whether one can relate the analysis of the stability of the constructed orbits to that of the $S_M(2,1,1)$ orbit, i.e., to analyze NR 2×2 matrices. Unfortunately, we cannot do that in general. We may follow Amritkar's way to analyze the stability of most the constructed orbits by analyzing $2N\times 2N$ matrices. But we are able to simplify the problem in some cases. For example, we can analyze the stability of the orbits $S_M(2N,s,1)$, $S_M(2N,L/2,R)$, and $S_M(2N,L-1,R)$ and all the random patterns in the GCML by dealing with

2×2 matrices instead of the $2NR\times2NR$ matrix or $2N\times2N$ matrices. We are involved in this problem at this moment.

Denoting

$$F_1^m=\left.\frac{df[x_n(i)]}{dx_n(i)}\right|_{x_n(i)=y_m},\quad F_2^m=\left.\frac{df[x_n(i)]}{dx_n(i)}\right|_{x_n(i)=z_m},\quad G_1^m=\left.\frac{dg[x_n(i)]}{dx_n(i)}\right|_{x_n(i)=y_m},\quad G_2^m=\left.\frac{dg[x_n(i)]}{dx_n(i)}\right|_{x_n(i)=z_m},\tag{8}$$

and the stability Jacobi matrix

$$J=J_1J_2\cdots J_{M-1}J_M\ ,\tag{9}$$

we first perform a permutation transformation on the Jacobi matrix J, i.e.,

$$J'=PJP^{-1}=PJ_1P^{-1}PJ_2P^{-1}\cdots PJ_{M-1}P^{-1}PJ_MP^{-1}=J_1'J_2'\cdots J_{M-1}'J_M'\ .\tag{10}$$

By properly choosing transformation matrix P, J_m' $(m=1,2,\ldots,M)$ can be cast into a block circulant matrix C with each block being a 2×2 matrix and may be written as

$$J_m'=bC(A_m,B_{1m},B_{2m},\ldots,B_{NR-1m})\ .\tag{11}$$

J_m' can be put into a block-diagonal form further by a unitary transformation [5,24]. Thus the block-diagonal form of the product matrix $J=J_1J_2\cdots J_{M-1}J_M$ is

$$J''=J_1''J_2''\cdots J_{M-1}''J_M''=\begin{bmatrix}\prod_{m=1}^{M}D_m^1 & 0 & \cdots & 0\\ 0 & \prod_{m=1}^{M}D_m^2 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & \prod_{m=1}^{M}D_m^{NR}\end{bmatrix},\tag{12}$$

where

$$D_m^{r+1}=A_m+\sum_{k=1}^{NR-1}\omega_r^kB_{km}\tag{13}$$

and

$$\omega_r^k=\exp(i2\pi rk/NR)=\exp(ik\theta)\ ,\quad r=0,1,2,\ldots,NR-1\ .\tag{14}$$

For $s=L/2$, we have

$$D_m^{r+1}=A_m-B_m+C_m\exp(ih\theta)\tag{15a}$$

for $r=1,2,\ldots,NR-1$, while

$$D_m^1=A_m+(NR-1)B_m+C_m\ ,\tag{15b}$$

where

$$A_m=\begin{bmatrix}F_1^m-\epsilon G_1^m & \frac{\epsilon}{2s}G_2^m\\ \frac{\epsilon}{2s}G_1^m & F_2^m-\epsilon G_2^m\end{bmatrix},\quad B_m=\begin{bmatrix}\frac{\epsilon}{2s}G_1^m & \frac{\epsilon}{2s}G_2^m\\ \frac{\epsilon}{2s}G_1^m & \frac{\epsilon}{2s}G_2^m\end{bmatrix},\tag{16}$$

and

$$C_m=\begin{cases}\begin{bmatrix}0 & \frac{\epsilon}{2s}G_2^m\\ \frac{\epsilon}{2s}G_1^m & 0\end{bmatrix} & \text{when } \beta=\frac{s+1}{2s}\\ \begin{bmatrix}\frac{\epsilon}{2s}G_1^m & 0\\ 0 & \frac{\epsilon}{2s}G_2^m\end{bmatrix} & \text{when } \beta=\frac{1}{2}\ .\end{cases}\tag{17}$$

Here $h=[(NR+1)/2]$ indicates the integer part. Therefore, to determine the stability of the orbits $S_M(2N,L/2,R)$ we only need to analyze the NR 2×2 matrices:

$$D(\theta)=\prod_{m=1}^{M}[A_m-B_m+C_m\exp(ih\theta)]\tag{18a}$$

for $\theta=2\pi/NR, 4\pi/NR,\ldots,2\pi(NR-1)/NR$, while

$$D(0)=\prod_{m=1}^{M}[A_m+(NR-1)B_m+C_m]=\prod_{m=1}^{M}\begin{bmatrix}F_1^m-\beta\epsilon G_1^m & \beta\epsilon G_2^m\\ \beta\epsilon G_1^m & F_2^m-\beta\epsilon G_2^m\end{bmatrix}.\tag{18b}$$

Similarly, for $S_M(2N,L-1,R)$ orbits and all the random patterns in the GCML, we get

$$D(\theta)=\prod_{m=1}^{M}(A_m-B_m)\tag{19a}$$

for $\theta=2\pi/NR, 4\pi/NR,\ldots,2\pi(NR-1)/NR$, while

$$D(0)=\prod_{m=1}^{M}[A_m+(NR-1)B_m]=\prod_{m=1}^{M}\begin{bmatrix}F_1^m-\beta\epsilon G_1^m & \beta\epsilon G_2^m\\ \beta\epsilon G_1^m & F_2^m-\beta\epsilon G_2^m\end{bmatrix},\tag{19b}$$

where

$$A_m = \begin{bmatrix} F_1^m - \epsilon G_1^m & \frac{\epsilon}{s} G_2^m \\ \frac{\epsilon}{s} G_1^m & F_2^m - \epsilon G_2^m \end{bmatrix} , \quad B_m = \begin{bmatrix} \frac{\epsilon}{s} G_1^m & \frac{\epsilon}{s} G_2^m \\ \frac{\epsilon}{s} G_1^m & \frac{\epsilon}{s} G_2^m \end{bmatrix} . \tag{20}$$

It is interesting to note that only four eigenvalues can be obtained no matter how large the phase space is. Denoting the two eigenvalues of the orbit $S_M(2,1,1)$ as $\lambda_1^0(\epsilon)$ and $\lambda_2^0(\epsilon)$, the two eigenvalues obtained from $D(0)$ for these orbits are just $\lambda_1^0(\beta\epsilon)$ and $\lambda_2^0(\beta\epsilon)$. The two eigenvalues obtained from $D(\theta)$ are

$$\lambda_1^1(\epsilon) = \prod_{m=1}^{M} (F_1^m - 2\beta\epsilon G_1^m) , \quad \lambda_2^1(\epsilon) = \prod_{m=1}^{M} (F_2^m - 2\beta\epsilon G_2^m) . \tag{21}$$

Because θ has $NR-1$ different values other than 0, the degeneracy of the states is at least $NR-1$. In addition, we can see from Eqs. (19) and (21) that all the orbits for a fixed system size L have the same stability property without considering their spatial configurations because $\beta = L/2(L-1)$ for all the orbits in the GCML.

For the $S_M(2N,s,1)$ orbits, the analysis is a little more complicated because the Jacobi matrix J would be different for different coupling length s. But we are able to reduce the problem to analyze N 2×2 matrices other than a $2N\times2N$ matrix. For example, we get for $s=N-1$

$$D(\theta) = \prod_{m=1}^{M} (A_m - B_m) \tag{22a}$$

for $\theta = 2\pi/N, 4\pi/N, \ldots, 2\pi(N-1)/N$, while

$$D(0) = \prod_{m=1}^{M} [A_m + (N-1)B_m] = \prod_{m=1}^{M} \begin{bmatrix} F_1^m - \beta\epsilon G_1^m & \beta\epsilon G_2^m \\ \beta\epsilon G_1^m & F_2^m - \beta\epsilon G_2^m \end{bmatrix} , \tag{22b}$$

where

$$A_m = \begin{bmatrix} F_1^m - \epsilon G_1^m & 0 \\ 0 & F_2^m - \epsilon G_2^m \end{bmatrix} , \quad B_m = \begin{bmatrix} \frac{\epsilon}{2s} G_1^m & \frac{\epsilon}{2s} G_2^m \\ \frac{\epsilon}{2s} G_1^m & \frac{\epsilon}{2s} G_2^m \end{bmatrix} . \tag{23}$$

This state is a degenerate state with degeneracy being at least $N-1$.

Although we cannot investigate the stability of all the $S_M(2N,s,R)$ orbits by investigating 2×2 matrices, one conclusion can be made that the stable regions of $S_M(2N,s,R)$ orbits cannot exceed the interval $(\epsilon_-/\beta, \epsilon_+/\beta)$, where (ϵ_-,ϵ_+) is the stable region of the orbit $S_M(2,1,1)$, because for all the constructed orbits we can get

$$D(0) = \prod_{m=1}^{M} \begin{bmatrix} F_1^m - \beta\epsilon G_1^m & \beta\epsilon G_2^m \\ \beta\epsilon G_1^m & F_2^m - \beta\epsilon G_2^m \end{bmatrix} . \tag{24}$$

The two eigenvalues are just $\lambda_1^0(\beta\epsilon)$ and $\lambda_2^0(\beta\epsilon)$. This indicates that the stability of the constructed orbits cannot be better than that of the $S_M(2,1,1)$ orbit, or, in other words, the orbits constructed from an unstable orbit must be unstable.

III. EXAMPLES

Here we consider a degenerate coupled logistic lattice of which $f(x)=g(x)=ax(1-x)$ as a concrete example. This model has been widely discussed by many authors [1,2,5–7]. Inserting $f(x)$ into Eq. (2), an $S_2(2,1,1)$ orbit is obtained:

$$y_1 = y_1(\epsilon) = \frac{1+a-2a\epsilon + [(1+a-2a\epsilon)^2 - 4(1-\epsilon)(1+a-2a\epsilon)]^{1/2}}{2a(1-2\epsilon)} ,$$
$$z_1 = z_1(\epsilon) = \frac{1+a-2a\epsilon - [(1+a-2a\epsilon)^2 - 4(1-\epsilon)(1+a-2a\epsilon)]^{1/2}}{2a(1-2\epsilon)} , \quad y_2(\epsilon) = z_1(\epsilon) \quad z_2(\epsilon) = y_1(\epsilon) , \tag{25}$$

which is an antiphase state. This orbit exists when $a \geq 3$, which is just the period doubling point of a single logistic map. The two eigenvalues of the stability Jacobi matrix are

$$\lambda_{1,2} = \frac{\epsilon}{1-2\epsilon} \pm \left[\frac{(2-3\epsilon)^2}{(1-2\epsilon)^2} + (1-2\epsilon)a(2-a) \right]^{1/2} \tag{26}$$

and the stability boundaries satisfy

$$\epsilon_- = \frac{2a(a-2)-3-\{[3+2a(a-2)]^2-4a(a-2)(a-1)^2\}^{1/2}}{4a(a-2)} ,$$
$$\epsilon_+ = \frac{1-[3/(a^2-2a)]^{1/2}}{2} . \quad (27)$$

Hopf bifurcation occurs at the lower bound ϵ_- and tangent bifurcation at the upper bound ϵ_+. As demonstrated by Amritkar *et al.* [5], the lower boundaries ϵ_- would be slightly different for orbits $S_2(2,1,R)$ with different replicas, and a wavelength doubling bifurcation is observed [6].

With the solution Eq. (25), we can predict that all the orbits belong to the special class. These predicted spatiotemporal orbits are all antiphase states or traveling waves with velocity N, except the random patterns in the GCML. As an example, we show two orbits of the GCML case in Fig. 1 ($L=30$).

The stability of an R replica solution can be, in general, reduced to the problem of illustrating R $2N\times 2N$ matrices [5]. But for the three cases discussed in Sec. II the stability of an orbit can be cast into the problem of illustrating NR 2×2 matrices, which can be analytically handled thoroughly. In the following we investigate analytically and numerically the stability of some of the orbits.

We first investigate the stability of the orbits in the GCML case in an analytical way. Using Eqs. (19b) and (21), the four eigenvalues obtained are

$$\lambda_{1,2}^0 = \frac{\beta\epsilon}{1-2\beta\epsilon} \pm \left[\frac{(2-3\beta\epsilon)^2}{(1-2\beta\epsilon)^2} + (1-2\beta\epsilon)a(2-a) \right]^{1/2} , \quad (28)$$

$$\lambda_{1,2}^1 = -a^2(1-2\beta\epsilon)^2 - 2a(1-2\beta\epsilon) + 4(1-\beta\epsilon)(1+a-2a\beta\epsilon) , \quad (29)$$

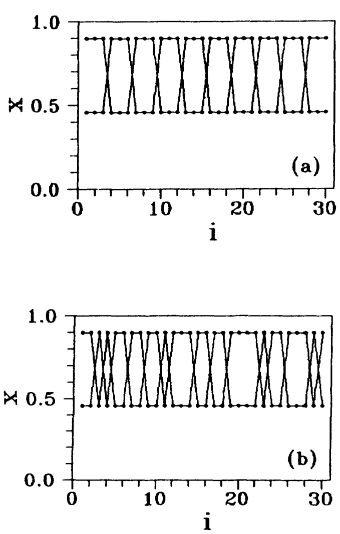

FIG. 1. Part of the predicted orbits in a coupled logistic GCML for $L=30$, $a=4$, and $\epsilon=0.15$. $\beta=\frac{15}{29}$. (a) $S_2(6,29,5)$; (b) random patterns.

where $\beta=L/2(L-1)$. These states are degenerate states with degeneracy $2(NR-1)$ other than $NR-1$. By a detailed study we find that for any system sizes the stability of these orbits is governed by Eq. (28), requiring $\epsilon\in[0,1]$, and thus the stability boundary is $(\epsilon_-/\beta, \epsilon_+/\beta)$. Hopf bifurcation occurs at ϵ_-/β and tangent bifurcation at ϵ_+/β. Equation (29) introduces no further conditions for the stability of these orbits in this ϵ interval.

Besides the case of the GCML, we have also numerically checked the stability of $S_2(4,1,R)$ and $S_2(4,2,R)$ orbits. We find no other conditions being introduced for the stability of these orbits. The stable regions of these orbits are just $[\epsilon_-/\beta, \epsilon_+/\beta]$, in which Hopf bifurcation occurs at ϵ_-/β and tangent bifurcation at ϵ_+/β. Here we also have $\epsilon\in[0,1]$.

IV. TWO-DIMENSIONAL CASE

In this section we only discuss the case of a nearest-neighbor coupled two-dimensional CML. The modeling equation is [4]

$$x_{n+1}(u,v) = f[x_n(u,v)] + \frac{\epsilon}{4}\{g[x_n(u-1,v)] + g[x_n(u+1,v)] + g[x_n(u,v-1)] + g[x_n(u,v+1)] - 4g[x_n(u,v)]\} . \quad (30)$$

The boundary condition is still assumed to be periodic and the system volume is $L_u L_v$. Many spatiotemporally periodic orbits can be obtained by analytical and numerical methods. Here we denote these orbits as $S_M(N_u, N_v; I, J)$ with N_u and N_v the periodicities in u and v directions, respectively. I and J are the replicas of the periodicity in the u and v directions, respectively, i.e., $I=L_u/N_u$ and $J=L_v/N_v$. Here we construct spatiotemporally periodic orbits from known orbits in the CML (2) instead of solving Eq. (30) analytically or numerically. For example, five spatial patterns can be formed from the $S_M(2,1,1)$ orbit, which are shown in Fig. 2. The state variables y_m and z_m also satisfy

$$z_{m+1} = f(z_m) + \beta\epsilon[g(y_m) - g(z_m)] . \quad (31)$$

A general case is that if there is an $S_M(N,1,R)$ orbit $(x_1^m x_2^m x_3^m \cdots x_N^m x_1^m \cdots x_N^m)$ in the CML (2) we can form two patterns in the CML (30): $S_M(1,N,I,J)$ and $S_M(N,N,I,J)$ (Fig. 3). The state variables satisfy

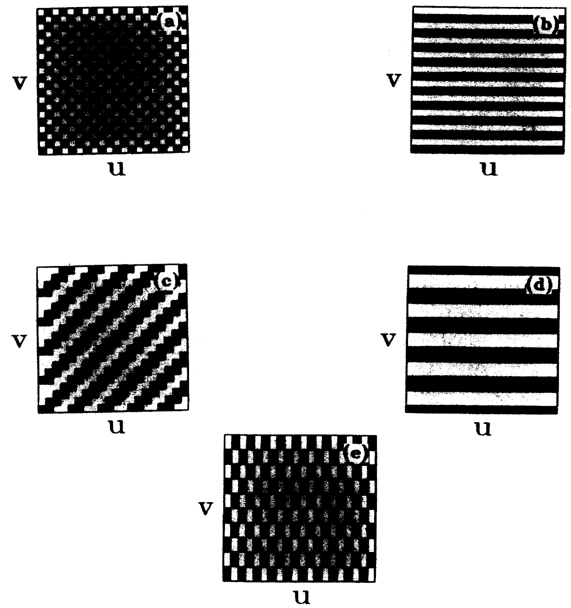

FIG. 2. Spatial patterns predicted from the $S_M(2,1,1)$ orbit. Pixels are black when $x_n(u,v)=y_m$ and white when $x_n(u,v)=z_m$. The system volume is 20×20. (a) $S_M(2,2;10,10)$, $\beta=1$; (b) $S_M(1,2;20,10)$, $\beta=\frac{1}{2}$; (c) $S_M(4,4;5,5)$, $\beta=\frac{1}{2}$; (d) $S_M(1,4;20,5)$, $\beta=\frac{1}{4}$; (e) $S_M(4,2;5,10)$, $\beta=\frac{3}{4}$.

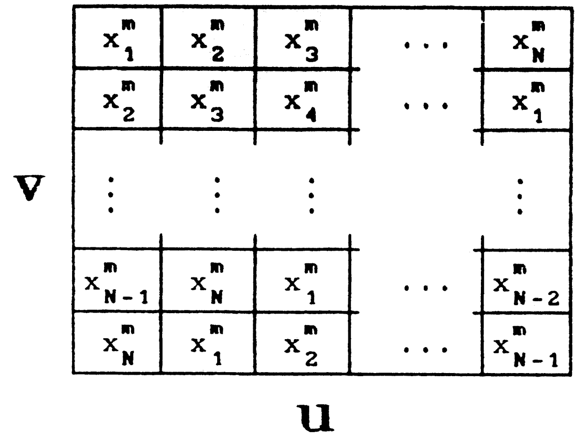

FIG. 3. $S_M(N,N;I,J)$ ($\beta=1$) orbit predicted from the $S_M(N,1,1)$ orbit. x_i^m ($i=1,2,\ldots,N$) represent the states of the lattice sites.

$$x_1^{m+1}=f(x_1^m)+\beta\epsilon[g(x_{i-1}^m)+g(x_{i+1}^m)-2g(x_i^m)] , \quad (32)$$

with $\beta=1$ and $\frac{1}{2}$ for $S_M(N,N;I,J)$ and $S_M(1,N;I,J)$, respectively. From this general result, it is reasonable that we can form, in the CML (30), two patterns from the $S_M(2,1,R)$ orbit [Figs. 2(a) and 2(b)] and two from the $S_M(4,1,R)$ orbit [Figs. 2(c) and 2(d)]. But due to the specialty of the spatial periodicity an additional pattern is constructed [Fig. 2(e)].

Although we can predict several patterns in the CML (30) from the orbits in the CML (2), there is still the question of how the stability of these patterns relates to those of the one-dimensional orbits. We are to investigate this problem below.

The stability of the $S_M(1,N,;I,J)$ orbits can be analyzed in the same manner as that carried out in Ref. 5. We only try to analyze the stability of the $S_M(N,N;I,J)$ orbits. We write down the variational form of the CML (30):

$$\delta x_{n+1}(u,v)=f'_{uv}\delta x_n(u,v)+\frac{\epsilon}{4}[g'_{u-1v}\delta x_n(u-1,v)+g'_{u+1v}\delta x_n(u+1,v)$$
$$+g'_{uv-1}\delta x_n(u,v-1)+g'_{uv+1}\delta x_n(u,v+1)-4g'_{uv}\delta x_n(u,v)] , \quad (33)$$

where we denote

$$f'_{uv}=\left.\frac{\partial f[x_n(u,v)]}{\partial x_n(u,v)}\right|_{x_n(u,v)=x_i^m}=F_i^m ,$$
$$g'_{uv}=\left.\frac{\partial g[x_n(u,v)]}{\partial x_n(u,v)}\right|_{x_n(u,v)=x_i^m}=G_i^m . \quad (34)$$

We denote all the state variables in the v direction for the same u as a vector:

$$\hat{X}_m(u)=\{x_m(u,1),x_m(u,2),\ldots,x_m(u,L_v)\} \quad (35)$$

and write Eq. (33) in the vector form as

$$\delta\hat{X}_{m+1}(u)=A_m^u\delta\hat{X}_m(u)+B_m^{u-1}\delta\hat{X}_m(u-1)$$
$$+B_m^{u+1}\delta\hat{X}_m(u+1) . \quad (36)$$

Looking at the fact that the vectors $\hat{X}_m(u)$, $u=1,2,\ldots,L_u$, have the following property:

$$\hat{X}_m(u)=\Gamma\hat{X}_m(u-1)=\Gamma^{u-1}\hat{X}_m(1) , \quad (37)$$

where

$$\Gamma=C(0,1,0,\ldots,0) , \quad (38)$$

we have

$$A_m^u=\Gamma^{-1}A_m^{u-1}\Gamma=\Gamma^{u-1}A_m^1\Gamma^{-u+1} ,$$
$$B_m^u=\Gamma^{-1}B_m^{u-1}\Gamma=\Gamma^{u-1}B_m^1\Gamma^{-u+1} . \quad (39)$$

We rewrite Eq. (36) as

$$\Gamma^{-u+1}\delta\hat{X}_m(u)=\Gamma^{-u+1}A_m\Gamma^{u-1}\Gamma^{-u+1}\delta\hat{X}_m(u)+\Gamma^{-u+1}B_m^{u-1}\Gamma^{u-2}\Gamma^{-u+2}\delta\hat{X}_m(u-1)+\Gamma^{-u+1}B_m^{u+1}\Gamma^{u}\Gamma^{-u}\delta\hat{X}_m(u+1)$$
$$=A_m^1\Gamma^{-u+1}\delta\hat{X}_m(u)+\Gamma B_m^1\Gamma^{-u+2}\delta\hat{X}_m(u-1)+\Gamma^{-1}B_m^1\Gamma^{-u}\delta\hat{X}_m(u+1) . \tag{40}$$

Thus it is reasonable to do a Fourier transformation, for Eq. (40), of the discrete series $\Gamma^{u-1}\delta\hat{X}_m(u)$ instead of $\delta\hat{X}_m(u)$ and we can reduce the problem of investigating the stability of the orbits $S_M(N,N;I,J)$ to that of analyzing the matrix

$$M(\theta)=\prod_{m=1}^{M}[A_m^1+\Gamma B_m^1 e^{i\theta}+\Gamma^{-1}B_m^1 e^{-i\theta}] , \tag{41}$$

where

$$\theta=0,\frac{2\pi}{L_u},\ldots,\frac{2(L_u-1)\pi}{L_u} \tag{42}$$

and A_m^1 and B_m^1 are block circulant matrices

$$A_m^1=b\mathrm{C}(A_m,C_m^+,0,\ldots,0,C_m^-) ,$$
$$B_m^1=b\mathrm{C}(B_m,0,0,\ldots,0,0) . \tag{43}$$

By this observation we obtain at once that $M(\theta)$ is still a circulant matrix and further simplification is possible. Finally we obtain

$$D(\theta,\phi)=\prod_{m=1}^{M}[A_m+C_m^-e^{i(\theta+2\phi)}+C_m^+e^{-i(\theta+2\phi)}+B_m^{-'}e^{i\theta}+B_m^{+'}e^{i\theta}] , \tag{44}$$

where A_m, C_m^-, C_m^+, $B_m^{-'}$, and $B_m^{+'}$ are

$$A_m=\begin{bmatrix} F_1^m-\epsilon G_1^m & \frac{\epsilon}{4}G_2^m & 0 & 0 & \cdots \\ \frac{\epsilon}{4}G_1^m & F_2^m-\epsilon G_2^m & \frac{\epsilon}{4}G_3^m & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

$$C_m^-=\begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{\epsilon}{4}G_N^m \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$C_m^+=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\epsilon}{4}G_1^m & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{45}$$

$$B_m^{-'}=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \frac{\epsilon}{4}G_1^m & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\epsilon}{4}G_{N-1}^m & 0 \end{bmatrix},$$

$$B_m^{+'}=\begin{bmatrix} 0 & \frac{\epsilon}{4}G_2^m & 0 & \cdots & 0 \\ 0 & 0 & \frac{\epsilon}{4}G_3^m & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and

$$\theta=\frac{2\pi p}{L_u} , \quad p=0,1,2,\ldots,L_u-1,$$
$$\phi=\frac{2\pi q}{J} , \quad q=0,1,2,\ldots,J-1 . \tag{46}$$

Therefore, we arrive at a very simple formula to analyze the stability of the constructed orbits by considering the $N\times N$ matrices instead of the $INJN\times INJN$ matrix or the $N^2\times N^2$ matrices argued in Ref. [5].

From the above discussion we can also conclude that the stability of the constructed orbits $S_M(1,N;I,J)$ and $S_M(N,N;I,J)$ cannot be better than that of the orbits $S_M(N,1,1)$ because $D(0,0)$ is just the stability Jacobi matrix of $S_M(N,1,1)$. In this section we only consider the two-dimensional CML in the case of nearest-neighbor coupling; things should be much more complicated for longer range couplings and an abundance of different spatial patterns might be constructed from the known one-dimensional orbits or the known two-dimensional orbits. The method of simplifying the stability analysis can be used to analyze other constructed orbits, which only need minor changes.

V. CONCLUSION

We have constructed a series of spatially periodic orbits in both one- and two-dimensional CMLs, from the known orbits of smaller phase space or smaller spatial periodicity, without directly solving the modeling equations. The stability of the constructed orbits can be analyzed by considering much smaller matrices; especially in the two-dimensional case the problem can be greatly simplified. From the analysis we reach a conclusion that the stability of the constructed orbits can never be better than that of the original ones, or, in other words, an orbit constructed from an unstable orbit is always unstable, but it is possible for an orbit constructed from a stable orbit to lose its stability by the enlargement of phase space, the changing of the strengths and lengths of couplings, or the changing of space configurations. In the one-dimensional case we can construct spatially periodic orbits only from the spatial period-2 orbits, but in the two-dimensional case, a spatial periodic orbit in a one-dimensional CML can always find its corresponding patterns in the two-dimensional case. In higher-dimensional symmetrically coupled CMLs, it is possible to construct many orbits from the orbits in a lower-dimensional CML and the discussion of the stability of the constructed orbits can be greatly simplified as we have demonstrated in Sec. IV. Finally, what we want to emphasize is that the models explored in this paper are coupled map lattices, but the same manipulations can be carried out in systems of coupled nonlinear oscillators. Thus our investigation can be suitable for a variety of physical systems.

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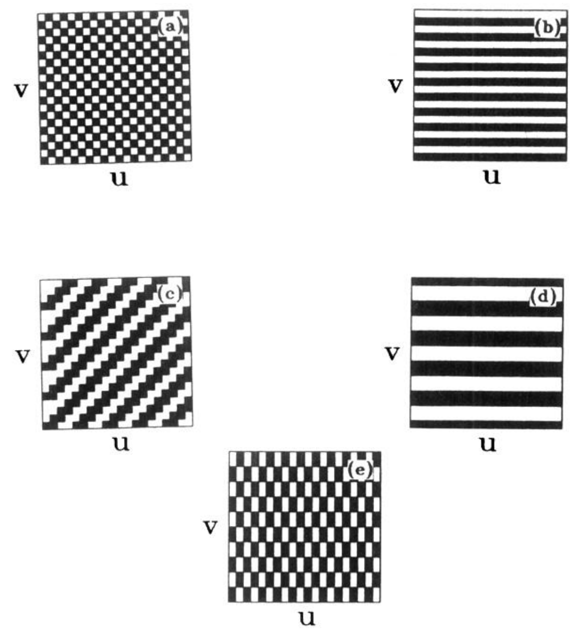

FIG. 2. Spatial patterns predicted from the $S_M(2,1,1)$ orbit. Pixels are black when $x_n(u,v)=y_m$ and white when $x_n(u,v)=z_m$. The system volume is 20×20. (a) $S_M(2,2;10,10)$, $\beta=1$; (b) $S_M(1,2;20,10)$, $\beta=\frac{1}{2}$; (c) $S_M(4,4;5,5)$, $\beta=\frac{1}{2}$; (d) $S_M(1,4;20,5)$, $\beta=\frac{1}{4}$; (e) $S_M(4,2;5,10)$, $\beta=\frac{3}{4}$.